Stability of jammed packings I: the rigidity length scale

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Abstract – In 2005, Wyart *et al.* (*Europhys. Lett.*, **72** (2005) 486) showed that the low frequency vibrational properties of jammed sphere packings can be understood in terms of a length scale, called ℓ^*, that diverges as the system becomes marginally unstable. Despite its tremendous success, it has been difficult to connect the counting argument that defines ℓ^* to other length scales that diverge near the jamming transition. We present an alternate derivation of ℓ^* and show that it arises naturally as the minimum size of a rigid cluster of particles with free boundaries. This understanding allows us to place a quantitative upper bound on the magnitude of ℓ^*. We also present the first direct numerical calculation of ℓ^*, and show that it is close to the upper bound.

Introduction. – Disordered solids exhibit many common features, including a characteristic temperature dependence of the heat capacity and thermal conductivity [1] and brittle response to mechanical load [2]. A rationalization for this commonality is provided by the jamming scenario [3,4], based on the behavior of ideal sphere packings, which exhibit a jamming transition with diverging length scales [3,5–12] as a function of packing fraction. According to the jamming scenario, these diverging length scales are responsible for commonality, much as a diverging length near a critical point is responsible for universality.

One key diverging length scale at the jamming transition arises from the so-called *cutting argument* introduced by Wyart *et al.* [6,7]. The cutting argument is a counting argument that compares the number of constraints on each particle to the number of degrees of freedom in a system with free boundary conditions. The cutting length, ℓ^*, is directly tied to the anomalous low-frequency behavior that leads to the distinctive heat capacity and thermal conductivity of disordered solids [1], and thus can be considered a cornerstone of our theoretical understanding of the jamming transition.

Despite its importance, however, the connection of the cutting length derived from counting arguments to physical length scales that diverge with the same exponent [5,9,10] has not been understood. In this paper, we calculate the cutting length directly and show that it has a simple physical interpretation as a rigidity length. This suggests that it might still be relevant even for systems for which counting arguments are less useful, such as packings of frictional particles [13–15] or ellipsoids [16–18]. Finally, we use the new interpretation of the cutting length as a rigidity scale to show that it is directly related to one of the length scales identified by Silbert *et al.* [5].

Review of the cutting argument [6,7]. – Consider an infinite, mechanically stable packing of frictionless spheres in d dimensions at zero temperature and applied stress. Two spheres repel if they overlap, *i.e.* if their center to center distance is less than the sum of their radii, but do not otherwise interact. "Rattler" particles that have no overlaps should be removed. Since the remaining degrees of freedom must be constrained, the average number of contacts on each particle, Z, must be greater than or equal to $2d$, which is precisely the jump in the contact number at the jamming transition [3,12]. All lengths will be given in units of σ, the average particle diameter, and frequencies will be given in units of $\sqrt{k_{\text{eff}}/m}$, where k_{eff} is the average effective spring constant of all overlapping particles and m is the average particle mass.

It is instructive to study a simpler system, the "unstressed" system, in which each repulsive interaction between pairs of particles in the system is replaced by a harmonic spring of stiffness k at its equilibrium length. The geometry of this spring network is identical to the geometry of the repulsive contacts between particles in

the original system and the vibrational properties of the two systems are closely related [7]. Now consider a square subsystem of linear size L obtained by removing all the contacts between particles (or, in the language of the unstressed system, all springs) that cross the boundary between the subsystem and the rest of the infinite system. Let the number of zero frequency modes in the cut system be q and the number of these *zero modes* that extend across the cut system be q'. Wyart *et al.* [6,7] used these modes to construct trial vibrational modes for the original infinite packing, as follows. If we restore the cut system with these q' extended zero modes back into the infinite system, the modes would no longer cost zero energy because of the contacts that connect the subsystem to the rest of the system. Suppose we deform each mode sinusoidally so that the amplitudes vanish at the boundary. This deformation increases the energy of each mode to order ω_L^2, where $\omega_L \sim 1/L$.

Note that if a mode is not extended, then it must be localized near the boundary, since the uncut system has no zero modes. However, the above procedure involves setting the mode amplitude to zero at the boundary, and so cannot be applied to such modes. It is therefore crucial to consider only the q' extended modes.

The cutting argument now makes the assumption that $q' = aq$, where a is a constant independent of L. Before the cut, the number of extra contacts in the subsystem above the minimum required for stability is $N_c^{\text{extra}} \sim (Z-2d)L^d$. When the cut is made, we lose $N_c^{\text{cut}} \sim L^{d-1}$ contacts. Naive constraint counting suggests that $q' \sim q = \max\left(-\left(N_c^{\text{extra}} - N_c^{\text{cut}}\right), 0\right)$, as shown by the solid black line in fig. 1. Since N_c^{extra} and N_c^{cut} both depend on L, we can define a length scale ℓ^* by

$$q' = 0 \qquad \text{if } L > \ell^* \tag{1}$$

$$q' > 0 \qquad \text{if } L < \ell^*. \tag{2}$$

The onset of zero modes is marked by $N_c^{\text{extra}} = N_c^{\text{cut}}$, so

$$\ell^* \sim \frac{1}{Z-2d}. \tag{3}$$

The variational argument predicts that at least $q'/2$ of the total L^d eigenmodes of the full system must have frequency less than ω_L, so the integral of the density of states from zero to ω_L must be

$$\int_0^{\omega_L} \mathrm{d}\omega D(\omega) \geq \frac{q'}{2L^d}. \tag{4}$$

However, $D(\omega)$ is an intrinsic property of the infinite system and must be independent of L. Therefore, assuming no additional low frequency modes beyond those predicted by the variational argument, we can vary L to back out the full density of states, as follows.

If $L > \ell^*$, then $q' = q = 0$ and

$$\int_0^{\omega_L} \mathrm{d}\omega D(\omega) = 0. \tag{5}$$

For $L < \ell^*$, we can write $q'/2 = a(N_c^{\text{cut}} - N_c^{\text{extra}})/2 = L^d(\omega_L - 1/\ell^*)$, where appropriate constants have been absorbed into ω_L and ℓ^*. This leads to

$$\int_0^{\omega_L} \mathrm{d}\omega D(\omega) = \omega_L - 1/\ell^*. \tag{6}$$

Equations (5) and (6) imply that

$$D(\omega) = \begin{cases} 0 & \text{if } \omega < \omega^* \\ \text{const.} & \text{if } \omega > \omega^*, \end{cases} \tag{7}$$

where $\omega^* \equiv 1/\ell^* \sim Z - 2d$ defines a frequency scale. Note that while ℓ^* is potential independent, the units of frequency, and thus ω^*, depend on potential [4]. This argument predicts that the density of states has a plateau that extends down to zero frequency at the jamming transition, where $Z - 2d = 0$. Above the jamming transition, when $Z - 2d > 0$, the plateau extends down to a frequency ω^* before vanishing. This agrees well with numerical results on the unstressed system [7]. Note the importance of the length scale ℓ^*, which defines the frequency scale ω^* and is responsible for the excess low frequency modes.

Need for a cleaner definition. – In the cutting argument, the length scale ℓ^* is defined as the size of a cut region, L, where the number of extended zero modes, q', first vanishes (eq. 2). The argument then assumes that this occurs when the cut system is isostatic, *i.e.* when $N_c^{\text{cut}} = N_c^{\text{extra}}$. Below, we briefly discuss a few complications that arise with this definition and motivate the need for a cleaner, more physical understanding of ℓ^*.

Shape of the cut. Wyart *et al.* only consider cutting along a flat surface and, while they argue that this is a reasonable choice for their variational argument, one can still imagine taking a cut of a different shape. If one were to consider a shape with a non-trivial fractal dimension, then N_c^{cut} would no longer scale as L^{d-1}, resulting in a length scale with entirely different scaling. Without some physical motivation for what cut to make, the scaling of ℓ^* is not well defined.

Dependence on the choice of degrees of freedom. The definition of ℓ^* in terms of the number of zero modes in eq. 2 is sensitive to the choice of degrees of freedom. For example, rattlers must be removed and internal degrees of freedom like particle rotations must be suppressed. For packings of ellipsoidal particles, to take one example, the choice of degrees of freedom is critical. Ellipsoids jam below isostaticity [16] and can have an extensive number of zero modes, suggesting that $\ell^* = \infty$ because cutting contacts can only increase the number of zero modes. However, when the aspect ratios of the ellipsoids are small, there is a band of modes similar to those for spheres, with a density of states that exhibits a plateau above $\omega^* \sim Z - 2d$ [17]. One would expect a length scale $\ell^* \sim 1/\omega^*$, but constraint counting fails to predict this.

Zero modes always exist at the boundary of a cut. The cutting argument assumes that the number of extended

zero frequency modes of the cut system, q', is proportional to the total number of zero modes of the cut system, q, with a proportionality constant a that does not depend on the size of the cut system, L. This assumption is critical to the derivation of the scaling of ℓ^*. In ref. [7], Wyart *et al.* show numerically that this is true for isostatic systems where $Z = 2d$, but they do not provide such evidence for over-constrained systems. We find that for $Z > 2d$, zero modes exist for all system sizes, implying that the relation $q' = aq$ does not always hold.

To demonstrate this, we generate packings of $N = 4096$ frictionless disks in 2 dimensions at zero temperature. Particles i and j interact with a harmonic, spherically symmetric, repulsive potential given by $V(r_{ij}) = \frac{\epsilon}{2}\left(1 - r_{ij}/\sigma_{ij}\right)^2$ only if $r_{ij} < \sigma_{ij}$, where r_{ij} is the center-to-center distance, σ_{ij} is the sum of their radii and $\epsilon \equiv 1$ sets the energy scale. Mechanically stable packings were generated with periodic boundary conditions by quenching the total energy to a local minimum (using a combination of linesearch methods, Newton's method and the FIRE algorithm [19]). To fix the distance to jamming, the density was adjusted until a target pressure, p, was reached. Systems were thrown out if the minimization algorithms did not converge. As in the cutting argument, we replace each packing with a geometrically equivalent unstressed spring network before measuring the vibrational modes.

We create a cut system by first periodically tiling the square unit cell, consistent with the periodic boundary conditions. We then remove all particles whose center is outside a box of length L. By first tiling the system, we are able to take cuts that are larger than the unit cell, as well as cuts that are smaller. We have checked that our results are not dependent on the choice of $N = 4096$ particles per unit cell.

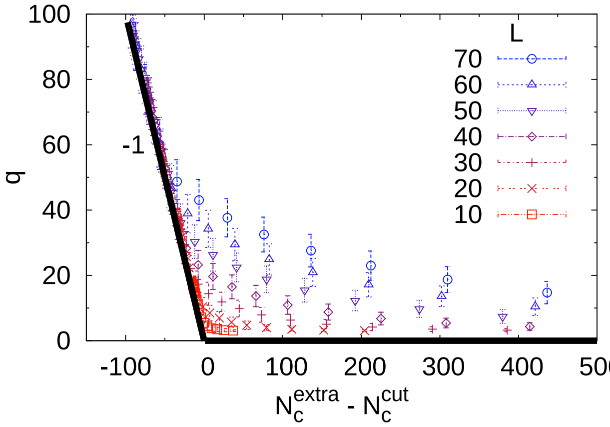

Fig. 1: Number of excess zero modes as a function of the number of excess contacts after the cut. Each data point is an average of configurations at constant pressure.

Figure 1 shows q as a function of $N_c^{\text{extra}} - N_c^{\text{cut}}$. Evidently, $q > 0$ for all cut sizes L and values of $N_c^{\text{extra}} - N_c^{\text{cut}}$. From the cutting argument, this would either imply $\ell^* = \infty$, or $q' \neq aq$. Note that these zero modes arise from locally under-constrained regions, despite the fact that the system is over constrained at the global level.

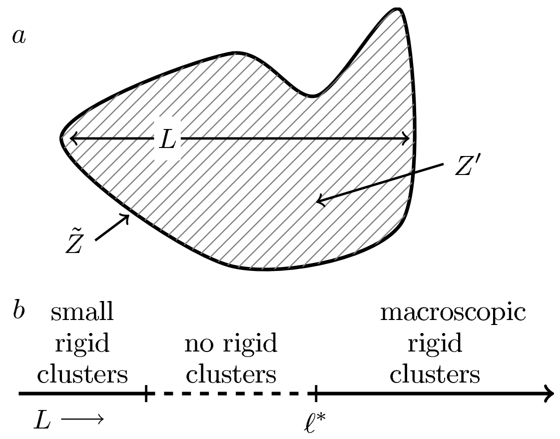

Fig. 2: a) An arbitrary surface (solid black line) of size L and an enclosed rigid cluster (stripes). The rigid cluster has an average contact number of Z' in the bulk and $\tilde{Z}$ at the boundary. As L becomes large, fluctuations in Z' and $\tilde{Z}$ vanish. b) Possible values of L such that a rigid cluster fits within the surface. Rigid clusters can either be small or larger than some minimum value. This minimum value defines ℓ^*.

The cutting argument can be rescued by assuming that the excess zero modes are associated with the boundary and that their number scales as L^{d-1} [6]. However, we find that these modes penetrate a non-negligible distance into the bulk of the system and that the scaling is not obvious.

Cluster argument. – We now reformulate the cutting argument in a way that does not rely on the *total* number of zero modes but is specifically designed to identify the onset of *extended* zero modes, which are the ones needed to obtain ω^*. The counting will be very similar to that in the cutting argument, but the setup and interpretation will be different. Our argument will be motivated by the simple fact that if none of the zero modes are extended, there must be a cluster of central particles that these modes cannot reach. Any deformation of this cluster costs energy, meaning it is rigid and has a non-zero bulk modulus. We will see that there is a strict lower bound for the size of such a rigid cluster, and that this minimum size defines ℓ^*. Thus, ℓ^* is the minimum size of a finite jammed system with free boundary conditions.

In our reformulation, we define a *rigid cluster* as a group of particles in an infinite d dimensional system with average contact number Z such that, if all other particles were removed, the only zero modes in the unstressed system would be those associated with global translation and rotation. This is purely a geometrical definition and is independent of potential. Now, consider an arbitrary $d-1$ dimensional closed surface with characteristic size L (for example, the solid black curve in fig. 2a). We will now ask whether or not it is *possible* for the cluster of particles within this surface to be rigid.

For the cluster to be rigid, it must satisfy

$$N_c - dN \geq -\frac{1}{2}d(d+1) \tag{8}$$

where N and N_c are the number of particles and contacts in the cluster, respectively, and $\frac{1}{2}d(d+1)$ is the number of global translations and rotations. This is a necessary but not sufficient condition for rigidity. We can write N_c as

$$N_c = \frac{1}{2}Z'(N - N_{\text{bndry}}) + \frac{1}{2}\tilde{Z}N_{\text{bndry}}, \tag{9}$$

where $\tilde{Z}$ is the contact number of the N_{bndry} particles on the boundary and Z' is the contact number of the particles not on the boundary (see fig. 2a). We can also define the positive constants a and b such that $N = 2aL^d$ and $N_{\text{bndry}} = 2bL^{d-1+\gamma}$, where $\gamma \geq 0$ depends on the shape of the surface, with $\gamma = 0$ for non-fractal shapes. For shapes that have multiple characteristic lengths, *e.g.* a long rectangle, the choice of which length to identify as L is irrelevant as it only leads to a change in the constants a and b. In practice, however, we will always take L to be the radius of gyration.

Equation (8) now becomes

$$aL^{d-1+\gamma}\left((Z'-2d)L^{1-\gamma} - c\right) \geq -\frac{1}{2}d(d+1), \tag{10}$$

where $c = \frac{b}{a}(Z' - \tilde{Z}) > 0$. Equation (10) is trivially satisfied if $(Z'-2d)L^{1-\gamma} - c > 0$, which implies

$$L > L_{\min}(Z', c, \gamma) \equiv \left(\frac{c}{Z'-2d}\right)^{1/(1-\gamma)}. \tag{11}$$

We will refer to clusters that satisfy eq. (11) as *macroscopic* clusters. However, it is also possible for $(Z' - 2d)L^{1-\gamma} - c < 0$, provided L is very small, because the right hand side of eq. (10) is small and negative.

It follows that it is only possible for the particles in our arbitrary surface to form a rigid cluster if the cluster is either very small or larger than $L_{\min}$; rigid clusters of intermediate sizes cannot exist!

Note that if L is large, then fluctuations in Z' and c vanish and $Z' = Z$. $L_{\min}$ is thus constant for all translations and rotations of the surface and is independent of L, depending only on the actual shape of the surface.

Given our arbitrary shape parameterized by c and γ, $L_{\min}(Z, c, \gamma)$ is the minimum possible size of any macroscopic rigid cluster in the $Z - 2d \ll 1$ limit. However, we wish to find the minimum size of any rigid cluster *regardless of shape*, which we do by finding c^* and γ^* that minimize $L_{\min}$ and defining $\ell^* \equiv L_{\min}(Z, c^*, \gamma^*)$. In the limit $Z \to 2d$, we immediately see that $\gamma^* = 0$ and

$$\ell^* = \frac{c^*}{Z-2d}. \tag{12}$$

As depicted in fig. 2b, we are left with the result that rigid clusters must either be very small or larger than ℓ^*.

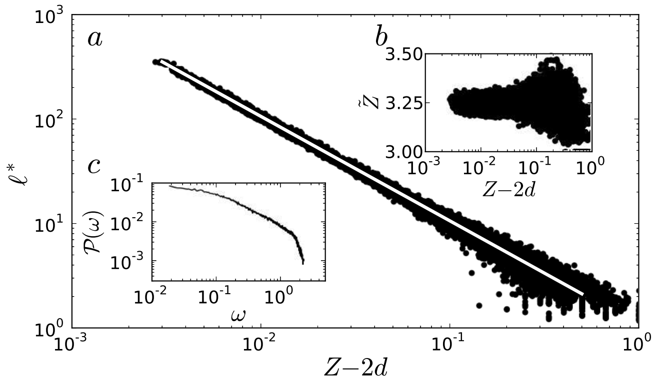

Fig. 4: a) ℓ^* as a function of $Z - 2d$, measured for individual systems as described in the text. b) $\tilde{Z} \approx 3.25$ in the limit $Z \to 2d$, close to the predicted bound. The solid white line in a) is the quantitative prediction of Eq. (13) using $\tilde{Z} = 3.25$. c) The projection, $\mathcal{P}(\omega)$, of the single extended zero mode just below ℓ^* onto the modes of frequency ω in the uncut system, averaged over many realizations.

We will now derive an upper bound for the magnitude of ℓ^* in the $Z \to 2d$ limit. Since $c \sim LN_{\text{bndry}}/N$, c is minimized when the shape is a d dimensional hypersphere. We can approximate N and N_{bndry} to be $N \approx \phi V_d^L$ and $N_{\text{bndry}} \approx \phi S_{d-1}^L$, where ϕ is the packing fraction and V_d^L and S_{d-1}^L are the volume and surface area of a d dimensional hypersphere with radius of gyration L. Using $S_{d-1}^L/V_d^L = w_d d/L$, where w_d is the ratio of the radius of gyration of a hypersphere to its radius[1], the $Z \to 2d$ limit of ℓ^* becomes

$$\ell^* \approx \frac{w_d d(2d - \tilde{Z})}{Z - 2d}. \tag{13}$$

Equation (13) is a quantitative derivation of ℓ^* as a function of Z that depends only on the value of $\tilde{Z}$, the average contact number at the boundary.

We put an upper bound on ℓ^* by obtaining a lower bound for $\tilde{Z}$. First, note that any particle at the boundary of the rigid cluster cannot have d or fewer contacts. Removing such a particle would remove d degrees of freedom and at most d constraints, and so the rigidity of the rest of the cluster would not be affected. Thus, $\tilde{Z} \geq d+1$ and

$$\ell^* \leq \frac{w_d d(d-1)}{Z-2d}. \tag{14}$$

Measuring ℓ^* from rigid clusters. – Note that not all surfaces larger than ℓ^* contain only a rigid cluster without non-trivial zero modes. Instead, there are almost always zero modes that exist around the boundary in particles that are not part of a rigid cluster. This is demonstrated in fig. 3, which shows progressively smaller cut systems in 2 dimensions. The faint blue disks are those

[1] $w_2 = \sqrt{1/2}$ and $w_3 = \sqrt{3/5}$

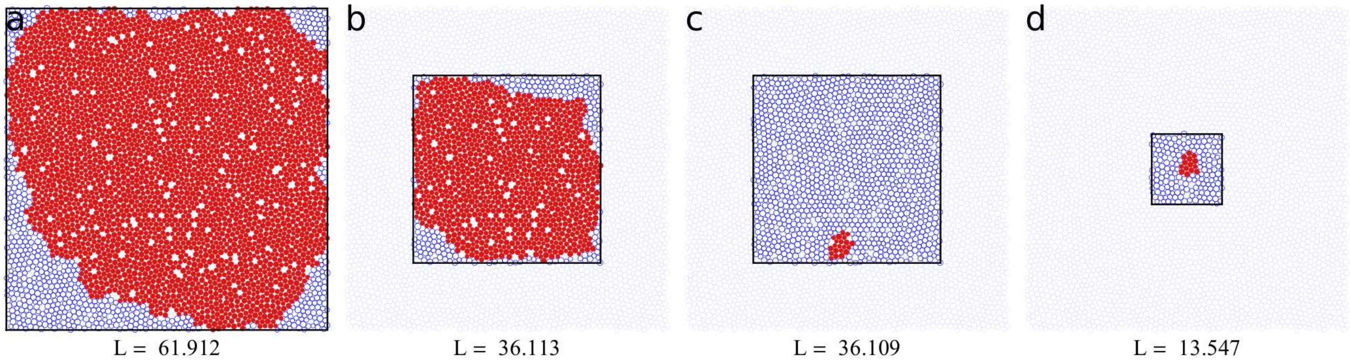

Fig. 3: Subsystems cut from a $N = 4096$ particle system. a) A large subsystem with $q = 60$ non-trivial zero modes. Only particles circled in blue participate in the zero modes. The solid red particles form a rigid cluster. b) A smaller subsystem with $q = 35$ zero modes. c) A subsystem obtained by removing one additional particle from the system in (b). This added a single additional zero mode that extends across the entire system. The largest remaining rigid cluster only contains 21 particles. **The appearance of this extended mode and the corresponding breakup of the rigid cluster marks the length scale ℓ^*.** d) A small system below ℓ^* with $q = 33$ zero modes. The largest rigid cluster contains 14 particles.

outside of the cut subsystem, while the bright blue circled disks are those that participate in zero-frequency modes in the cut subsystem. Particles that do not participate in any zero mode are shown in red. These red disks form the largest rigid cluster that fits into the square subsystem.

To identify the particles that do not participate in zero modes, we use a pebble game algorithm that was developed in refs. [20, 21] to understand the rigidity percolation transition in bond- and site-diluted lattices. This algorithm decomposes any network into distinct rigid clusters and can also be used to calculate the number of zero modes.

Figure 3 illustrates the qualitative picture presented in fig. 2b. In fig. 3a, there is a rigid cluster that covers about 84% of the system. In fig. 3b, a smaller subsystem is cut from the same original system. In this smaller subsystem, the largest rigid cluster is smaller than in fig. 3a, but likewise covers about 84% of the subsystem. In fig. 3c, the cut subsystem has been reduced by just enough so it contains one less particle than fig. 3b. However, the macroscopic rigid cluster has vanished and the largest rigid cluster in fig. 3c contains only 21 particles! As the size of the cut region is decreased further (fig. 3d), the largest rigid cluster remains of order 10 particles.

We emphasize that the breakup of the rigid cluster between figs. 3b and 3c is sharp: removing one particle completely destroys the macroscopic rigid cluster with no intermediate sized rigid clusters, as predicted above. However, the cluster in fig. 3b is not the smallest macroscopic cluster because there still may be particles at the boundary of the cluster that are not *essential* for rigidity. To find the smallest macroscopic cluster, we pick a particle at the boundary and remove it only if doing so does not break apart the cluster. We do this until all particles at the boundary have been checked, and then measure the radius of gyration of the cluster, which we identify as ℓ^*.

Figure 4a shows that ℓ^* diverges as $(Z - 2d)^{-1}$, consistent with the cutting argument and our reformulation. Figure 4b shows that $\tilde{Z} \approx 3.25$ as $Z \to 2d$, close to the theoretical lower bound of 3. The solid white line in fig. 4 shows the quantitative prediction from eq. (13) using $\tilde{Z} = 3.25$, which agrees extremely well with the data.

According to ref. [6], the extended zero modes of the cut system should be good trial modes for the low frequency modes of the system with periodic boundaries. Consider a system just below ℓ^* so that there is only one extended zero mode. The global translations and rotations, as well as the boundary zero modes, can be projected out of the set of zero modes by comparing them to the modes of the system just above ℓ^*. Figure 4c shows the projection of that single extended zero mode onto the dN modes of the full uncut system as a function of the frequency of the uncut modes. This mode projects most strongly onto the lowest frequency modes, implying that it is, in fact, a good trial mode from which to extract the low frequency behavior, as assumed [6].

Discussion. – We have presented an alternative definition of the length scale ℓ^* from a cluster argument based on rigidity. In contrast to the cutting argument, we did not need to assume anything about the number of zero modes in a cut system and we considered all possible cuts simultaneously. As in the cutting argument, however, we did assume that spatial fluctuations in Z are negligible. Wyart *et al.* argue [7] that fluctuations in Z are negligible in $d > 2$ dimensions, and that the condition of local force balance suppresses such fluctuations even in $d = 2$ in jammed packings. We have applied our procedure from the previous section to bond-diluted hexagonal lattices where these fluctuations are not suppressed. Although these systems display a global rigidity transition [20, 21] when they have periodic boundary conditions, they do not exhibit an abrupt loss of rigidity at some length scale that could be interpreted as ℓ^* when they have free boundary

conditions. It remains to be seen if ℓ^* exists in this sense for bond-diluted 3 dimensional lattices.

The interpretation of ℓ^* presented here still depends on the choice of degrees of freedom. It also relies on knowledge of the vibrational modes of the unstressed system, which is difficult to obtain experimentally. However, our result that ℓ^* marks a rigidity transition suggests that the elastic properties of a system could be used to measure ℓ^*. Such a measurement should be experimentally tractable, would not require knowledge of the vibrational properties, and would not require specification of the degrees of freedom of the system.

We note that the definition of the length scale ℓ^* as the size of the smallest macroscopic cluster that can be rigid, or equivalently, that can have nonzero elastic moduli, follows naturally from the notion of jamming as the onset of rigidity, or more precisely, of nonzero elastic moduli [12]. The rigidity interpretation of ℓ^* makes it transparently clear that the cutting length ℓ^* is equivalent to the length scale, ℓ_{L}, identified by Silbert *et al.* [5]. For systems with periodic boundaries, the anomalous modes derived from the zero modes swamp out sound modes at frequencies above ω^*. Thus, the minimum wavelength of longitudinal sound that can be observed in the system is $\ell_{\mathrm{L}} = c_{\mathrm{L}}/\omega^*$, where $c_{\mathrm{L}} = \sqrt{B/\rho}$ is the longitudinal speed of sound, B is the bulk modulus, and ρ is the mass density of the system.

For systems with free boundaries, rigid clusters cannot exist on length scales below ℓ^*, so the bulk modulus and speed of sound vanish. The minimum wavelength of longitudinal sound that can be supported is therefore given by the minimum macroscopic cluster size, ℓ^*. From the scalings of B and ω^*, we see that $\ell_{\mathrm{L}} \sim (Z - 2d)^{-1} \sim \ell^*$. Our definition of ℓ^* implies that the two length scales not only have the same scaling but have the same physical meaning.

Silbert *et al.* also identified a second smaller length scale ℓ_{T} from the transverse speed of sound, which depends on the shear modulus. For systems with free boundaries to be rigid, they must support both longitudinal and transverse sound, and so while our reasoning applies to both ℓ_{L} and ℓ_{T}, ℓ^* should be the larger of the two, so that the condition for rigidity for a cluster of size L is $L \gtrsim \ell^* = \ell_{\mathrm{L}}$. Note that systems with periodic boundary conditions of size $L \gg \ell_{\mathrm{T}}$ are stable to infinitesimal deformations of the shape of the boundary [22, 23].

Our result that rigid clusters cannot exist on length scales below ℓ^* appears to be consistent with results of Tighe [24], as well as that of Düring *et al.* [25], for floppy networks below isostaticity. There, they find that clusters with free boundaries replaced by pinned boundaries cannot be rigid for length scales above $1/|Z - 2d|$. The use of pinning boundary particles has also been used by Mailman and Chakraborty [26] to calculate a point-to-set correlation length above the transition that appears to scale as ℓ^*.

Ideal sphere packings have the special property that the number of contacts in a packing with periodic boundary conditions is exactly isostatic at the jamming transition in the thermodynamic limit [3,12]. Here, we have shown that the number of contacts in such a system with *free* boundary conditions is exactly isostatic (eq. (8) is satisfied with a strict equality) in the cluster of size ℓ^*. This simplicity makes ideal sphere packings a uniquely powerful model for exploring the marginally jammed state.

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REFERENCES

[1] Phillips W. A., (Editor) *Amorphous Solids: Low-Temperature Properties* (Springer-Verlag) 1981.
[2] Maloney C. E. and Robbins M. O., *Chaos*, **17** (2007) 041105.
[3] O'Hern C. S., Silbert L. E., Liu A. J. and Nagel S. R., *Phys. Rev. E*, **68** (2003) 011306.
[4] Liu A. J. and Nagel S. R., *Annu. Rev. Condens. Matter Phys.*, **1** (2010) 347.
[5] Silbert L. E., Liu A. J. and Nagel S. R., *Phys. Rev. Lett.*, **95** (2005) 098301.
[6] Wyart M., Nagel S. R. and Witten T. A., *EPL*, **72** (2005) 486.
[7] Wyart M., Silbert L. E., Nagel S. R. and Witten T. A., *Phys. Rev. E*, **72** (2005) 051306.
[8] Drocco J., Hastings M., Olson Reichhardt C. J. and Reichhardt C., *Phys. Rev. Lett.*, **95** (2005) 088001.
[9] Ellenbroek W. G., Somfai E., van Hecke M. and van Saarloos W., *Phys. Rev. Lett.*, **97** (2006) 258001.
[10] Ellenbroek W. G., van Hecke M. and van Saarloos W., *Phys. Rev. E*, **80** (2009) 061307.
[11] Vågberg D., Valdez-Balderas D., Moore M., Olsson P. and Teitel S., *Phys. Rev. E*, **83** (2011) 030303.
[12] Goodrich C. P., Liu A. J. and Nagel S. R., *Phys. Rev. Lett.*, **109** (2012) 095704.
[13] Somfai E., van Hecke M., Ellenbroek W. G., Shundyak K. and van Saarloos W., *Phys. Rev. E*, **75** (2007) 020301.
[14] Shundyak K., van Hecke M. and van Saarloos W., *Phys. Rev. E*, **75** (2007) 010301.
[15] Henkes S., van Hecke M. and van Saarloos W., *EPL*, **90** (2010) 14003.
[16] Donev A., Connelly R., Stillinger F. H. and Torquato S., *Phys. Rev. E*, **75** (2007) 051304.
[17] Zeravcic Z., Xu N., Liu A. J., Nagel S. R. and van Saarloos W., *EPL*, **87** (2009) 26001.
[18] Mailman M., Schreck C. F., O'Hern C. S. and Chakraborty B., *Phys. Rev. Lett.*, **102** (2009) 255501.

[19] Bitzek E., Koskinen P., Gähler F., Moseler M. and Gumbsch P., *Phys. Rev. Lett.*, **97** (2006) 170201.
[20] Jacobs D. J. and Thorpe M. F., *Phys. Rev. Lett.*, **75** (1995) 4051.
[21] Jacobs D. J. and Thorpe M. F., *Phys. Rev. E*, **53** (1996) 3682.
[22] Schoenholz S. S., Goodrich C. P., Kogan O., Liu A. J. and Nagel S. R., arXiv:1301.6982 [**cond-mat.soft**].
[23] Dagois-Bohy S., Tighe B., Simon J., Henkes S. and van Hecke M., *Phys. Rev. Lett.*, **109** (2012) 095703.
[24] Tighe B., *Phys. Rev. Lett.*, **109** (2012) 168303.
[25] During G., Lerner E. and Wyart M., *Soft Matter*, **9** (2012) 146.
[26] Mailman M. and Chakraborty B., *J. Stat. Mech.*, **2011** (2011) L07002.